Issue : XII,Vol. - I
V R J F P S
IMPACT FACTOR 6.30
ISSN 2348-7976
Jan. 2019 To June 2019

(28)

9

A Study on the Greatest Eigenvalue of Distance Matrix of the Wheel Graph

Mahesh S. Wavare
Dept. of Mathematics,
Rajarshi Shahu Mahavidyalaya,
Latur, Dist. Latur

Dr. J.K.Mane
Dept. of Mathematics,
Sanjeevani Mahavidyalaya,
Chapoli, Dist. Latur

Anil Thenge
Dept. of Mathematics,
Shivaji Mahavidyalaya,
Renapur, Dist. Latur

Research Paper -Mathematics

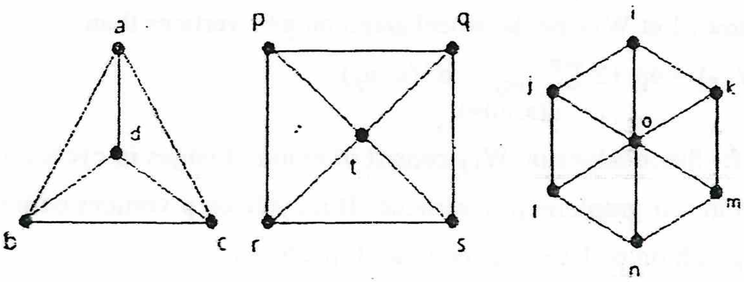
ABSTRACT

The distance energy of the graph is the sum of absolute eigenvalues of the distance matrix of the graph .in this paper we obtain the upper and lower bound for the distance energy of the wheel graph as well as the upper bound and loer bound for greatest eigenvalue of distance matrix of wheel graph .

Keywords : energy of graph , distance energy , eigenvalue , eigenvector , wheel graph .

Introduction :

In this paper we consider the wheel graph W1,P . Wheel graph is obtain from Cp by joining each vertex to new vertex v is the wheel on p+1 vertices .



I(W_4) II(W_5) III(W_7)

The concept of energy was introduced by I. Gutman in the 1978. [1] .let G be the graph on n vertices then adjacency matrix of graph G is denoted as the A(G) and is defined as

$A(G) = [a_{ij}]_{n\times n}$

Where $a_{ij}= \begin{cases} 1 \text{ if } v_i \text{ is adjacent to } v_j \\ 0 \quad \text{otherwise} \end{cases}$

Let $\lambda_1,\lambda_2,\ldots\lambda_n$ be the eigenvalues of the A(G) . The energy of the graph is defined as the sum of absolute eigenvalues of the adjacency matrix of the graph .

i.e. $E(G) = \sum_{i-1}^{n} |\lambda_i|$ for more details see [2,3]

The distance matrix of the graph G is denoted as the $A_D(G)$ and is define as

$A_D(G) = [d_{ij}]_{n\times n}$

Where $d_{ij} = \begin{cases} \text{distance of } v_i \text{ to } v_j \\ 0 \text{ if } v_i \text{ is not connected to } v_j \end{cases}$

Let $\mu_1, \mu_2, \mu_3, \ldots\ldots, \mu_n$ be the eigenvalues of the distance matrix $A_D(G)$ of G then distance energy $E_D(G)$ of G is define as

$E_D(G) = \sum_{i=1}^{n} |\mu_i|$ for details on distance energy see [4,5] . motivated by this concept of distance energy we compute the upper bound and lower bound of distance energy of wheel graph also we established the best possible upper bound and lower bound for the greatest eigenvalue of the wheel graph

Lemma : Let $W_{1,P}$ be the wheel graph on p+1 vertices then

$E_D(W_{1,P}) = 4p + 2 \sum^{p}_{\substack{i<j \\ d(u_i,u_j)\neq 1}} d^2(u_i,u_j)$

Proof:- Since edges in $W_{1,P}$ consist of union of edges in cycle on p vertices and edges in star graph on p+1 vertices . But cycle on p vertices contain p edges and star graph on p+1 vertices consist of p edges .

Therefore number of edges in $W_{1,P}$ = p + p = 2p .

Since the sum of squares of eigenvalues of $A_D(W_{1,P})$ is the trace of $A_D(W_{1,P})^2$.

$\therefore E_D(G) = \sum_{i=0}^{p} \mu_i^2$

$$= \sum_{i=0}^{p} \sum_{j=0}^{p} d_{ij} d_{ji}$$

$$= \sum_{i=0}^{p} d_{ii}^2 + \sum_{\substack{i=0 \\ i \neq j}}^{p} d_{ij} d_{ji}$$

$= \sum_{\substack{i=0 \\ i \neq j}}^{p} d_{ij} d_{ji}$ since $d_{ii} = 0$ for the wheel graph .

$= 2\sum_{\substack{i=0 \\ i<j}}^{p} d(u_i, u_j)^2$

$\because$ In the distance matrix of the wheel graph there are p edges (from the above discussion)suct that$d(u_i, u_j)$=1 .

$= 2[\, 2p + \sum_{\substack{i=0, \\ i<j, d(v_i, v_j) \neq 1}}^{p} d(u_i, u_j)^2]$

$$= 4p + 2\sum_{\substack{i=0 \\ i<j, d(v_i, v_j) \neq 1}}^{p} d(u_i, u_j)^2$$

$\therefore E_D(W_{1,P}) = 4p + 2\sum_{\substack{i<j \\ d(u_i, u_j) \neq 1}}^{p} d^2(u_i, u_j)$

Theorem : Let G be the wheel graph $W_{1,P}$, then

$$\sqrt{4p + 2\sum_{\substack{i<j \\ d(u_i, u_j) \neq 1}}^{p} d^2(u_i, u_j)} \leq E_D(W_{1,P}) \leq \sqrt{(p+1)\left[4p + 2\sum_{\substack{i<j \\ d(u_i, u_j) \neq 1}}^{p} d^2(u_i, u_j)\right]}$$

Proof : consider ,

$$(E_D(G))^2 = \left(\sum_{i=0}^{p} |\mu_i|\right)^2$$

Using statement of Cauchy- Schwarz inequality

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

Take $x_i = 1$ and $y_i = |\mu_i|$

We get ,

$\left(\sum_{i=0}^{p} |\mu_i|\right)^2 \leq \left(\sum_{i=0}^{p} 1\right)\left(\sum_{i=0}^{p} \mu_i^2\right)$

$$= (P+1)\left[4p + 2\sum_{\substack{i<j \\ d(u_i,u_j)=1}}^{p} d^2(u_i,u_j)\right]$$

$$\therefore (E_D(G)) \leq \sqrt{(P+1)\left[4p + 2\sum_{\substack{i<j \\ d(u_i,u_j)\neq 1}}^{p} d^2(u_i,u_j)\right]} \quad (1)$$

Now , consider the inequality ,

$$\left(\sum_{i=0}^{p} |x_i|\right)^2 \geq \sum_{i=0}^{p} x_i^2$$

$$\therefore \left(\sum_{i-0}^{p} |\mu_i|\right)^2 \geq \sum_{i-0}^{p} \mu_i^2$$

$$\therefore (E_D(G))^2 \geq \left[4p + 2\sum_{\substack{i<j \\ d(u_i,u_j)\neq 1}}^{p} d^2(u_i,u_j)\right]$$

Taking square root on both side

$$\therefore E_D(G) \geq \sqrt{\left[4p + 2\sum_{\substack{i<j \\ d(u_i,u_j)\neq 1}}^{p} d^2(u_i,u_j)\right]} \quad (2)$$

From above two equations ,

$$\sqrt{\left[4p + 2\sum_{\substack{i<j \\ d(u_i,u_j)\neq 1}}^{p} d^2(u_i,u_j)\right]} \leq E_D(G) \leq \sqrt{(p+1)\left[4p + 2\sum_{\substack{i<j \\ d(u_i,u_j)\neq 1}}^{p} d^2(u_i,u_j)\right]}$$

Theorem :-Let G be a wheel graph and $\mu(G)$ is the greatest eigenvalue of the matrix then $\mu(G) \geq (p-3) + \sqrt{p^2 - 3p + 9}$.

Proof :- Let $V = \{ v_0 , v_1 , v_2 ,\ldots\ldots, v_p\}$,

Consider the distance matrix of G .

$$D(G) = \begin{bmatrix} d_{00} & d_{01} & \cdots\cdots\cdots & d_{0p} \\ d_{10} & d_{11} & \cdots\cdots\cdots & d_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ d_{p0} & d_{p1} & \cdots\cdots\cdots & d_{pp} \end{bmatrix}$$

From above two equations ,

$$\sqrt{\left[4p + 2\sum_{\substack{i<j \\ d(u_i,u_j)\neq 1}}^{p} d^2(u_i,u_j)\right]} \leq E_D(G) \leq \sqrt{(p+1)\left[4p + 2\sum_{\substack{i<j \\ d(u_i,u_j)\neq 1}}^{p} d^2(u_i,u_j)\right]}$$

Theorem :- Let G be a wheel graph and $\mu(G)$ is the greatest eigenvalue of the matrix then $\mu(G) \geq (p-3) + \sqrt{p^2 - 3p + 9}$.

Proof :- Let $V = \{ v_0 , v_1 , v_2 ,........, v_p\}$,

Consider the distance matrix of G .

$$D(G) = \begin{bmatrix} d_{00} & d_{01} & .... & d_{0p} \\ d_{10} & d_{11} & .... & d_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ d_{p0} & d_{p1} & .... & d_{pp} \end{bmatrix}$$

Let $X = (x_0 , x_1, x_2, ... x_p)^T$ be the eigenvector of $D(G)$ corresponding to the greatest eigenvalue $\mu(G)$.

$\therefore D(G)\, X = \mu(G)\, X$

$$\begin{bmatrix} d_{00} & d_{01} & .... & d_{0p} \\ d_{10} & d_{11} & .... & d_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ d_{p0} & d_{p1} & .... & d_{pp} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_p \end{bmatrix} = \mu(G) \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_p \end{bmatrix} \text{------------(1)}$$

Since the graph G is wheel graph , diameter of G is 1 or 2 . i.e.$d_{ij} = 0$ or $d_{ij} = 1$ or $d_{ij} = 2$.

Let $A = \{ v_0\}$ and $B = \{ v_1 , v_2 ,........, v_p \}$

Take x_i= minx_k and x_j = min x_k , $k > 0 x_k \in X$.

From equation (1)

$\mu(G)\, x_i = \sum_{k=0}^{p} d_{0k} x_k$

$\therefore \mu(G)\, x_i \geq p\, x_i$ ($\because \sum_{k=0}^{p} d_{0k} = 1$ and x_i is min of x_k) ----------(2) .

Also

$\mu(G)\, x_j = \sum_{k=n}^{p} d_{jk} x_k$

$\therefore \mu(G)\, x_j = \sum_{d_{jk}-1} d_{jk} x_k + \sum_{d_{jk}-2} d_{jk} x_k$ ---------------(3)

Since each row of D(G) contains only three 1's , $\therefore \sum_{d_{jk}=1} d_{jk} = 3$.

And each row of D(G) contains (p-3) number of 2's , $\therefore \sum_{d_{jk}-2} d_{jk} x_k = 2(p-3)$

From equation (3),

$\mu(G)\, x_j \geq 3\, x_i + 2(p-3)\, x_j$

$\therefore \mu(G) x_j - 2(p-3)\, x_j \geq 3\, x_i$

$\therefore (\mu(G) - 2(p-3))\, x_j \geq 3\, x_i$ --------(4)

$\therefore \mu(G)\,(\mu(G) - 2(p-3))\, x_i x_j \geq 3p\, x_i x_j$

$\therefore \mu(G)\,(\mu(G) - 2(p-3)) \geq 3p$

$\therefore \mu^2(G) - 2(p-3)\mu(G) - 3p \geq 0$

Solving this quadratic equation , we get .

$\therefore \mu(G) \geq \frac{2(p-3) \pm \sqrt{4(p-3)^2 - 4(-3p)}}{2}$

$= (p-3) \pm \sqrt{x^2 - 6p + 9 + 3p}$.

$= (p-3) \pm \sqrt{x^2 - 3p + 9}$

$\therefore \mu(G) \geq (p-3) + \sqrt{x^2 - 3p + 9}$

Theorem :- Let G be a wheel graph and $\mu(G)$ is the greatest eigenvalue of the matrix then $\mu(G) \leq d(p-3) + \sqrt{d^2p^2 - 3d^2p + 9d^2}$. where d is the diameter of the wheel graph .

Proof :- Let V = { v_0 , v_1 , v_2 ,........, v_p} ,

Consider the distance matrix of G .

$$= \begin{bmatrix} d_{00} & d_{01} & \dots\dots\dots & d_{0p} \\ d_{10} & d_{11} & \dots\dots\dots & d_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ d_{p0} & d_{p1} & \dots\dots\dots & d_{pp} \end{bmatrix}$$

Let $X = (x_0, x_1, x_2, \dots\dots\dots x_p)^T$ be the eigenvector of $D(G)$ corresponding to the greatest eigenvalue $\mu(G)$.

$\therefore D(G)\,X = \mu(G)\,X$

$$\begin{bmatrix} d_{00} & d_{01} & \dots\dots\dots & d_{0p} \\ d_{10} & d_{11} & \dots\dots\dots & d_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ d_{p0} & d_{p1} & \dots\dots\dots & d_{pp} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_p \end{bmatrix} = \mu(G) \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_p \end{bmatrix} \text{-----------(4)}$$

Let $A = \{ v_0 \}$ and $B = \{ v_1, v_2, \dots\dots, v_p \}$

Take $x_i = \max x_k$ and $x_j = \max x_k$, $k > 0$ $x_k \in X$.

From equation (1)

$\mu(G)\, x_i = \sum_{k=0}^{p} d_{0k} x_k$

Since d is diameter of G , d > 0 .

$\therefore \mu(G)\, x_i \le d\,p\,x_i$ ($\because \sum_{k=0}^{p} d_{0k} = 1$ and x_i is max of x_k) -----------(2)

Also

$\mu(G)\, x_j = \sum_{k=0}^{p} d_{jk} x_k$

$\therefore \mu(G)\, x_j = \sum_{d_{jk}=1} d_{jk} x_k + \sum_{d_{jk}=2} d_{jk} x_k$ ---------------(5)

$\mu(G)\, x_j \le 3d\, x_i + 2d(p-3)\, x_j$

$\therefore \mu(G)\, x_j - 2d(p-3)\, x_j \le 3d\, x_i$

$\therefore (\mu(G) - 2d(p-3))\, x_j \le 3\,d\, x_i$ --------(4)

$\therefore \mu(G)\,(\mu(G) - 2\,d(p-3))\, x_i x_j \le 3dp\, x_i x_j$

$\therefore \mu(G)\,(\mu(G) - 2d(p-3)) \le 3dp$

$\therefore \mu^2(G) - 2d(p-3)\mu(G) - 3dp \leq 0$

$\therefore \mu(G) \leq \frac{2d(p-3) \pm \sqrt{4d^2(p-3)^2 - 4(-3dp)}}{2}$

$= d(p-3) \pm \sqrt{d^2x^2 - 6d^2p + 9d^2 + 3d^2p}$

$= d(p-3) \pm \sqrt{d^2p^2 - 3d^2p + 9d^2}$

$\therefore \mu(G) \leq d(p-3) + \sqrt{d^2p^2 - 3d^2p + 9d^2}$ ------------(6)

We get the required result .

Remark :The lower and upper bound of greatest eigenvalue of wheel graph obtain in above theorem are equal when $G = W_{1,p}$ where $P < 4$.

References :-

[1] I.Gutman the energy of graph .ber.math.Statist.sekt.forschungzgraz.103 (1978),1-22.

[2] R.B.Bapat Graph and Matrices Hindustan book agency 2011.

[3] X.Li.Y.shi.&gutman graph energy springer , New York Heidelberg Dordrechet London 2012.

[4] G caporossi. E. chasset and B. Furtla , some conjecture and properties on the distance energy . lescachiers du GERAD, 64(2009) , 1.7.

[5] A.D. Gungor and S.B. Bozkurt , on the distance spectral radius and distance energy of wheel graph , linear multilinear algebra 59(2011) (365-370)

[6] G Indulal , sharp bounds on the distance spectral radius and the distance energy of graph , linear Algebra appl. 430 (2009) 106 113 .

[7] B.Zhou , on the largest eigenvalue of distance matrix of a tree MATCH commun.math.comput.chem 13 (2008) 123 131 .

[8] Kinkarch.Das , on the largest eigenvalue of the distance matrix of bipartite graph MATCH commun.math.comput.chem 62(2009) 667-672 .

[9] K.Balsubramanian , computer generation of distance polynomial of graph , J.comput chem. 11 (1990) 829-836 .

[10] AleksandarIli , Distance spectra and distance energy of integral circulant Graphs , faculty of science and mathematics , Visegradska 33 , 18000 Nis ,Serbia

[11] On distance energy of graphs GopalapillaiIndulal,aIvan Gutmanband Ambat Vijayakumarc